Singular Solutions of Hessian Elliptic Equations in Five Dimensions

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1 Introduction

In this paper we study a class of fully nonlinear second-order elliptic equations of the form

$$F(D^2u) = 0 \tag{1.1}$$

defined in a domain of $\mathbb{R}^n$. Here D^2u denotes the Hessian of the function u. We assume that F is a Lipschitz function defined on the space $S^2(\mathbb{R}^n)$ of $n \times n$ symmetric matrices satisfying the uniform ellipticity condition, i.e. there exists a constant $C = C(F) \geq 1$ (called an *ellipticity constant*) such that

$$C^{-1}||N|| \leq F(M+N) - F(M) \leq C||N|| \tag{1.2}$$

for any non-negative definite symmetric matrix N; if $F \in C^1(S^2(\mathbb{R}^n))$ then this condition is equivalent to

$$\frac{1}{C'}|\xi|^2 \leq F_{u_{ij}}\xi_i\xi_j \leq C'|\xi|^2 \ , \forall \xi \in \mathbb{R}^n \ . \tag{1.2'}$$

Here, u_{ij} denotes the partial derivative $\partial^2 u/\partial x_i \partial x_j$. A function u is called a *classical* solution of (1) if $u \in C^2(\Omega)$ and u satisfies (1.1). Actually, any classical solution of (1.1) is a smooth ($C^{\alpha+3}$) solution, provided that F is a smooth (C^α) function of its arguments.

For a matrix $S \in S^2(\mathbb{R}^n)$ we denote by $\lambda(S) = \{\lambda_i : \lambda_1 \leq ... \leq \lambda_n\} \in \mathbb{R}^n$ the (ordered) set of eigenvalues of the matrix S. Equation (1.1) is called a Hessian equation ([T1],[T2] cf. [CNS]) if the function $F(S)$ depends only on the eigenvalues $\lambda(S)$ of the matrix S, i.e., if

$$F(S) = f(\lambda(S)),$$

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for some function f on $\mathbb{R}^n$ invariant under permutations of the coordinates.

In other words the equation (1.1) is called Hessian if it is invariant under the action of the group $O(n)$ on $S^2(\mathbb{R}^n)$:

$$\forall O \in O(n),\ F({}^tO \cdot S \cdot O) = F(S)\ . \tag{1.3}$$

The Hessian invariance relation (1.3) implies the following:

(a) F is a smooth (real-analytic) function of its arguments if and only if f is a smooth (real-analytic) function.

(b) Inequalities (1.2) are equivalent to the inequalities

$$\frac{\mu}{C_0} \leq f(\lambda_i + \mu) - f(\lambda_i) \leq C_0\mu,\ \forall \mu \geq 0,$$

$\forall i = 1, ..., n$, for some positive constant C_0.

(c) F is a concave function if and only if f is concave.

Well known examples of the Hessian equations are Laplace, Monge-Ampère, Bellman, Isaacs and Special Lagrangian equations.

Bellman and Isaacs equations appear in the theory of controlled diffusion processes, see [F]. Both are fully nonlinear uniformly elliptic equations of the form (1.1). The Bellman equation is concave in $D^2u \in S^2(\mathbb{R}^n)$ variables. However, Isaacs operators are, in general, neither concave nor convex. In a simple homogeneous form the Isaacs equation can be written as follows:

$$F(D^2u) = \sup_b \inf_a L_{ab}u = 0, \tag{1.4}$$

where L_{ab} is a family of linear uniformly elliptic operators of type

$$L = \sum a_{ij} \frac{\partial^2}{\partial x_i \partial x_j} \tag{1.5}$$

with an ellipticity constant $C > 0$ which depends on two parameters a, b.

Consider the Dirichlet problem

$$\begin{cases} F(D^2u, Du, u, x) = 0 & \text{in } \Omega \\ \qquad u = \varphi & \text{on } \partial\Omega\ , \end{cases} \tag{1}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain with a smooth boundary $\partial\Omega$ and φ is a continuous function on $\partial\Omega$.

We are interested in the problem of existence and regularity of solutions to the Dirichlet problem (1.6) for Hessian equations and Isaacs equation. The problem (1.6) has always a unique viscosity (weak) solution for fully nonlinear elliptic equations (not necessarily Hessian equations). The viscosity solutions satisfy the equation (1.1) in a weak sense, and the best known interior regularity ([C],[CC],[T3]) for them is $C^{1+\epsilon}$ for some $\epsilon > 0$. For more details see

[CC], [CIL]. Until recently it remained unclear whether non-smooth viscosity solutions exist. In the recent papers [NV1], [NV2], [NV3], [NV4] the authors first proved the existence of non-classical viscosity solutions to a fully nonlinear elliptic equation, and then of singular solutions to Hessian uniformly elliptic equation in all dimensions beginning from 12. Those papers use the functions

$$w_{12,\delta}(x) = \frac{P_{12}(x)}{|x|^{\delta}},\ w_{24,\delta}(x) = \frac{P_{24}(x)}{|x|^{\delta}},\ \delta \in [1,2[,$$

with $P_{12}(x), P_{24}(x)$ being cubic forms as follows:

$$P_{12}(x) = Re(q_1 q_2 q_3),\ x = (q_1, q_2, q_3) \in \mathbb{H}^3 = \mathbb{R}^{12},$$

$\mathbb{H}$ being Hamiltonian quaternions,

$$P_{24}(x) = Re((o_1 \cdot o_2) \cdot o_3) = Re(o_1 \cdot (o_2 \cdot o_3)),\ x = (o_1, o_2, o_3) \in \mathbb{O}^3 = \mathbb{R}^{24}$$

$\mathbb{O}$ being the algebra of Caley octonions.

Finally, the paper [NTV] gives a construction of non-smooth viscosity solution in 5 dimensions which is order 2 homogeneous, also for Hessian equations, the function

$$w_5(x) = \frac{P_5(x)}{|x|},$$

being such solution for the Cartan minimal cubic

$$P_5(x) = x_1^3 + \frac{3x_1}{2}\left(z_1^2 + z_2^2 - 2z_3^2 - 2x_2^2\right) + \frac{3\sqrt{3}}{2}\left(x_2 z_1^2 - x_2 z_2^2 + 2z_1 z_2 z_3\right)$$

in 5 dimensions.

However, the methods of [NTV] does not work for the function $w_{5,\delta}(x) = P_5(x)/|x|^{\delta},\ \delta > 1$, and thus does not give singular (i.e. not in $C^{1,1}$) viscosity solutions to fully nonlinear equations in 5 dimensions.

In the present paper we fill the gap and prove

Theorem 1.1.
The function

$$w_{5,\delta}(x) = P_5(x)/|x|^{1+\delta},\ \delta \in [0,1[$$

is a viscosity solution to a uniformly elliptic Hessian equation (1.1) *with a smooth functional F in a unit ball $B \subset \mathbb{R}^5$ for the isoparametric Cartan cubic form*

$$P_5(x) = x_1^3 + \frac{3x_1}{2}\left(z_1^2 + z_2^2 - 2z_3^2 - 2x_2^2\right) + \frac{3\sqrt{3}}{2}\left(x_2 z_1^2 - x_2 z_2^2 + 2z_1 z_2 z_3\right)$$

with $x = (x_1, x_2, z_1, z_2, z_3)$.

In particular one gets the optimality of the interior $C^{1,\alpha}$-regularity of viscosity solutions to fully nonlinear equations in dimensions 5 and more; note also

that all previous constructions give only Lipschitz Hessian functional F. Let us recall that in the paper [NV5] it is proven that there is no order 2 homogenous solutions to elliptic equations in 4 dimensions which suggests strongly that in 4 (and less) dimensions there is no homogenous non-classical solutions to uniformly elliptic equations.

As in [NV3] we get also that $w_{5,\delta}(x),\ \delta \in [0,1[$ is a viscosity solution to a uniformly elliptic Isaacs equation:

Corollary 1.2.
The function

$$w_{5,\delta}(x) = P_5(x)/|x|^{1+\delta},\ \delta \in [0,1[$$

is a viscosity solution to a uniformly elliptic Isaacs equation (1.4) *in a unit ball* $B \subset \mathbb{R}^5$.

The rest of the paper is organized as follows: in Section 2 we recall some necessary preliminary results and we prove our main results in Section 3. The proof in Section 3 extensively uses MAPLE but is completely rigorous.

2 Preliminary results

Let $w = w_n$ be an odd homogeneous function of order $2-\delta$, $0 \le \delta < 1$, defined on a unit ball $B = B_1 \subset \mathbb{R}^n$ and smooth in $B \setminus \{0\}$. Then the Hessian of w is homogeneous of order $-\delta$.

Define the map

$$\Lambda : B \longrightarrow \lambda(S) \in \mathbb{R}^n \ .$$

$\lambda(S) = \{\lambda_i : \lambda_1 \le ... \le \lambda_n\} \in \mathbb{R}^n$ being the (ordered) set of eigenvalues of the matrix $S = D^2 w$. Denote Σ_n the permutation group of $\{1, ..., n\}$. For any $\sigma \in \Sigma_n$, let T_σ be the linear transformation of $\mathbb{R}^n$ given by $x_i \mapsto x_{\sigma(i)},\ i = 1, ..., n$.

Let $a, b \in B$. Denote by $\mu_1(a,b) \le ... \le \mu_n(a,b)$ the eigenvalues of $(D^2w(a) - D^2w(b))$.

Lemma 2.1. *Assume that for a smooth function $g : U \longrightarrow \mathbb{R}$ where the domain U contains*

$$M := \bigcup_{\sigma \in \Sigma_n} T_\sigma \Lambda(B) \subset \mathbb{R}^n$$

one has

$$g_{|M} = 0.$$

Assume also the condition

$$\min_{i=1,...,5} \inf_{x \in M} \left\{ \frac{\partial g}{\partial \lambda_i}(\lambda) \right\} > 0. \qquad (2.1)$$

Assume further that for any $a, b \in B$ either $\mu_1(a,b) = ... = \mu_n(a,b) = 0$ or

$$1/C \le -\frac{\mu_1(a,b)}{\mu_n(a,b)} \le C, \tag{2.2}$$

where C is a positive constant (may be, depending on M, g but not on a, b). If $\delta > 0$ we assume additionally that w changes sign in B. Then w is a viscosity solution in B of a uniformly elliptic Hessian equation (1.1) with a smooth F. Function w is as well a solution to a uniformly elliptic Isaacs equation.

Proof. Denote for any $\theta > 0$ by $K_\theta \subset \mathbb{R}^n$ the cone $\{\lambda \in \mathbb{R}^n,\ \lambda_i/|\lambda| > \theta\}$, and let K_θ^* be its dual cone. Let x, y be orthogonal coordinates in $\mathbb{R}^n$ such that $x = \lambda_1 + ... + \lambda_n$ and y be the orthogonal complement of x. Denote by p the orthogonal projection of $\mathbb{R}^n$ on subspace y. Denote

$$\Gamma = \{g = 0\} \subset U,$$

$$G = p(\Gamma),$$

$$m = p(M).$$

From (2.1), (2.2) it follows that the surface Γ is a graph of a smooth function h defined on G. By k_θ we denote the function on y which graph is the surface ∂K_θ^*. We define the function $H(y)$ by

$$H(y) = \inf_{z \in G} \{h(z) + k_\theta(y - z)\}.$$

We fix a sufficiently small $\theta > 0$. Then from (2.1), (2.2) it follows that $H = h$ on G. Denote by J the graph of H. It is easy to show, see similar argument in [NV1], [NV3], that for any $a, b \in J$, $a \neq b$,

$$1/C \le -\min_i(a_i - b_i)/\max_i(a_i - b_i) \le C. \tag{2.3}$$

Let E be a smooth function in $\mathbb{R}^{n-1}$ with the support in a unit ball and with the integral being equal to 1. Denote $E_c(y) = c^{-n+1}E(y/c)$, $c > 0$. Set

$$H_c = H * E_c.$$

Then H_c will be a smooth function such that any two points a, b on its graph will satisfy (2.3). Moreover $H_c \to H$ in $C(\mathbb{R}^n)$ as c goes to 0, and $H_c \to h$ in C^∞ on compact subdomains of G. Thus for a sufficiently small $c > 0$ we can easily modify function H_c to a function $\widetilde{H}$ such that $\widetilde{H}$ will coincide with h in a neighborhood of m, coincide with H in the complement of G and the points on the graph of $\widetilde{H}$ will still satisfy (2.3) possibly with a larger constant C. Define the function F in $\mathbb{R}^n$ by

$$F = x - \widetilde{H}(y).$$

Then w is a solution in $\mathbb{R}^n \setminus \{0\}$ of a uniformly elliptic Hessian equation (1.1) with such defined nonlinearity F. As in [NV3], [NV4] it follows that w is a

viscosity solution of (1.1) in the whole space $\mathbb{R}^n$. In [NV3] we have shown that the equation (1.1) for the function w can be rewritten in the form of the Isaacs equation. The lemma is proved.

We will apply this result to the function $w_{5,\delta}(x) = P_5(x)/|x|^{1+\delta}$.

Let then recall some facts from [NTV] about the Cartan cubic form $P_5(x)$.

Lemma 2.2.
The form $P_5(x)$ admits a three-dimensional automorphism group.

Indeed, one easily verifies that the orthogonal trasformations

$$A_1(\phi) := \begin{pmatrix} \frac{3\cos(\phi)^2-1}{2} & \frac{\sqrt{3}\sin(\phi)^2}{2} & 0 & 0 & \frac{\sqrt{3}\sin(2\phi)}{2} \\ \frac{\sqrt{3}\sin(\phi)^2}{2} & \frac{1+\cos(\phi)^2}{2} & 0 & 0 & \frac{-\sin(2\phi)}{2} \\ 0 & 0 & \cos(\phi) & \sin(\phi) & 0 \\ 0 & 0 & -\sin(\phi) & \cos(\phi) & 0 \\ \frac{-\sqrt{3}\sin(2\phi)}{2} & \frac{\sin(2\phi)}{2} & 0 & 0 & \cos(2\phi) \end{pmatrix}$$

$$A_2(\psi) := \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & \cos(2\psi) & 0 & -\sin(2\psi) & 0 \\ 0 & 0 & \cos(\psi) & 0 & -\sin(\psi) \\ 0 & \sin(2\psi) & 0 & \cos(2\psi) & 0 \\ 0 & 0 & \sin(\psi) & 0 & \cos(\psi) \end{pmatrix}$$

$$A_3(\theta) := \begin{pmatrix} \frac{3\cos(\theta)^2-1}{2} & \frac{-\sqrt{3}\sin(\theta)^2}{2} & 0 & 0 & \frac{-\sqrt{3}\sin(2\theta)}{2} \\ \frac{-\sqrt{3}\sin(\theta)^2}{2} & \frac{1+\cos(\theta)^2}{2} & 0 & 0 & \frac{-\sin(2\theta)}{2} \\ 0 & 0 & \cos(\theta) & -\sin(\theta) & 0 \\ 0 & 0 & \sin(\theta) & \cos(\theta) & 0 \\ \frac{\sqrt{3}\sin(2\theta)}{2} & \frac{\sin(2\theta)}{2} & 0 & 0 & \cos(2\theta) \end{pmatrix}$$

do not change the value of $P_5(x)$.

Lemma 2.3.
Let G_P be subgroup of $SO(5)$ generated by $\{A_1(\phi), A_2(\psi), A_3(\theta) : (\phi, \psi, \theta) \in \mathbb{R}^3\}$. Then the orbit $G_P S^1$ of the circle

$$S^1 = \{(\cos(\chi), 0, \sin(\chi), 0, 0) : \chi \in \mathbb{R}\} \subset S^4$$

under the natural action of G_P is the whole S^4.

We need also the following two simple algebraic results ([NV3, Lemmas 2.2 and 4.1]):

Lemma 2.4. *Let A, B be two real symmetric matrices with the eigenvalues $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$ and $\lambda_1' \geq \lambda_2' \geq \ldots \geq \lambda_n'$ respectively. Then for the eigenvalues $\Lambda_1 \geq \Lambda_2 \geq \ldots \geq \Lambda_n$ of the matrix $A - B$ we have*

$$\Lambda_1 \geq \max_{i=1,\cdots,n} (\lambda_i - \lambda_i'), \;\; \Lambda_n \leq \min_{i=1,\cdots,n} (\lambda_i - \lambda_i').$$

Lemma 2.5. *Let $\delta \in [0,1)$, W, $\bar{W} \in \mathbb{R}$ with $|W| \leq \frac{1}{3\sqrt{3}}, |\bar{W}| \leq \frac{1}{3\sqrt{3}}$ and let $\mu_1(\delta) \geq \mu_2(\delta) \geq \mu_3(\delta)$ (resp., $\bar{\mu}_1(\delta) \geq \bar{\mu}_2(\delta) \geq \bar{\mu}_3(\delta)$) be the roots of the polynomial*

$$T^3 + 3W(1+\delta)T^2 + (3W^2(1+\delta)^2 - 1)T + W(1-\delta) + W^3(1+\delta)^3$$

(*resp. of the polynomial*

$$T^3 + 3\bar{W}(1+\delta)T^2 + (3\bar{W}^2(1+\delta)^2 - 1)T + \bar{W}(1-\delta) + \bar{W}^3(1+\delta)^3 \text{).}$$

Then for any $K > 0$ verifying $|K-1| + |\bar{W} - W| \neq 0$ one has

$$\frac{1-\delta}{5+\delta} =: \rho \leq \frac{\mu_+(K)}{-\mu_-(K)} \leq \frac{1}{\rho} = \frac{5+\delta}{1-\delta}$$

where

$$\mu_-(K) := \min\{\mu_1(\delta) - K\bar{\mu}_1(\delta),\ \mu_2(\delta) - K\bar{\mu}_2(\delta),\ \mu_3(\delta) - K\bar{\mu}_3(\delta)\},$$

$$\mu_+(K) := \max\{\mu_1(\delta) - K\bar{\mu}_1(\delta),\ \mu_2(\delta) - K\bar{\mu}_2(\delta),\ \mu_3(\delta) - K\bar{\mu}_3(\delta)\}\ .$$

3 Proofs

Let $w_{5,\delta} = P_5/|x|^{1+\delta}$, $\delta \in [0,1[$. By Lemma 2.1 it is sufficient to prove the existence of a smooth function g verifying the conditions (2.1) and (2.2). We beging with calculating the eigenvalues of $D^2 w_{5,\delta}(x)$. More precisely, we need

Lemma 3.1.
Let $x \in S^4$, and let $x \in G_P(p,0,q,0,0)$ with $p^2 + q^2 = 1$. Then

$$Spec(D^2 w_{5,\delta}(x)) = \{\mu_{1,\delta}, \mu_{2,\delta}, \mu_{3,\delta}, \mu_{4,\delta}, \mu_{5,\delta}\}$$

for

$$\mu_{1,\delta} = \frac{p(p^2\delta + 6 - 3\delta)}{2},$$

$$\mu_{2,\delta} = \frac{p(p^2\delta - 3 - 3\delta) + 3\sqrt{12 - 3p^2}}{2},$$

$$\mu_{3,\delta} = \frac{p(p^2\delta - 3 - 3\delta) - 3\sqrt{12 - 3p^2}}{2},$$

$$\mu_{4,\delta} = -\frac{p\delta(6-\delta)(3-p^2) + \sqrt{D(p,\delta)}}{4},$$

$$\mu_{5,\delta} = -\frac{p\delta(6-\delta)(3-p^2) - \sqrt{D(p,\delta)}}{4},$$

and

$$D(p,\delta) := (6-\delta)(4-\delta)(2-\delta)\delta(p^2-3)^2 p^2 + 144(\delta-2)^2 > 0.$$

The characteristic polynomial $F(S)$ of $D^2 w$ is given by

$$F(S) = S^5 + a_{1,\delta} S^4 + a_{2,\delta} S^3 + a_{3,\delta} S^2 + a_{4,\delta} S + a_{5,\delta}$$

for

$$\begin{aligned}
a_{1,\delta} &= \frac{(\delta+1)(\delta-8)b}{2}, \\
a_{2,\delta} &= \frac{(\delta+1)(21\delta+13-4\delta^2)b^2}{4} + 9(2\delta-\delta^2-4), \\
a_{3,\delta} &= \frac{(6\delta^2-31\delta-1)(\delta+1)^2 b^3}{8} + \frac{27(4\delta-2\delta^2+5+\delta^3)}{2}, \\
a_{4,\delta} &= \frac{(2\delta-1)(5-\delta)(\delta+1)^2 b^4}{8} + \frac{9(\delta-1)(\delta^2-2\delta+9)}{2}, \\
a_{5,\delta} &= \frac{b(1-\delta)\left(b^2(\delta+1)^3+108(1-\delta)\right)\left(b^2(\delta+1)(\delta-5)+36(\delta-1)\right)}{32}, \\
& \textit{where} \quad b := p(p^2-3).
\end{aligned}$$

Note that the spectrum in this lemma is unordered one.

Proof of Lemma 3.1. Since $w_{5,\delta}$ is invariant under G_P, we can suppose that $x = (p, 0, q, 0, 0)$. Then $w_{5,\delta}(x) = \frac{p(3-p^2)}{2}$ and we get by a brute force calculation:

$$D^2 w_{5,\delta}(x) := \begin{pmatrix} M_{1,\delta} & 0 \\ 0 & M_{2,\delta} \end{pmatrix}$$

being a block matrix with

$$M_{1,\delta} := \frac{1}{2} \begin{pmatrix} m_{1,1} & m_{1,2} & m_{1,3} \\ m_{1,2} & m_{2,2} & m_{2,3} \\ m_{1,3} & m_{2,3} & m_{3,3} \end{pmatrix},$$

$$\begin{aligned}
m_{1,1} &:= -(\delta+2)\delta p^5 + (\delta+3)\delta p^3 + (12-9\delta)p, \\
m_{1,2} &:= 3\sqrt{3}p(p^2-1)\delta, \\
m_{1,3} &:= -q\left((\delta+2)\delta p^4 + 3\delta(1-\delta)p^2 + 3\delta - 6)\right) \\
m_{2,2} &:= \delta p^3 - 3(\delta+4)p, \\
m_{2,3} &:= 3\sqrt{3}q(\delta p^2 + 2 - \delta), \\
m_{3,3} &:= (\delta+2)\delta p^5 + (5-4\delta)\delta p^3 - 3(\delta-1)(2-\delta)p,
\end{aligned}$$

$$M_{2,\delta} := \frac{1}{2} \begin{pmatrix} \delta p^3 + 3(2-\delta)p & 6\sqrt{3}q \\ 6\sqrt{3}q & \delta p^3 - 3(4+\delta)p \end{pmatrix}$$

which gives for the characteristic polynomial $F(S) = F_1(S) \cdot F_2(S) \cdot F_3(S)$ where

$$F_1(S) := S - \frac{p(p^2\delta + 6 - 3\delta)}{2};$$

$$F_2(S) = S^2+\frac{\delta p(p^2-3)(\delta-6)S}{2}+\frac{(2-\delta)\left((\delta-6)\delta p^6+6(6-\delta)\delta^2p^4+9(\delta^2-6\delta)p^2+36(\delta-2)\right)}{4};$$

$$F_3(S) := S^2+(3+3\delta-\delta p^2)pS+\frac{(p^2-3)(\delta^2p^4-3\delta^2p^2-6\delta p^2+36)}{4};$$

and the spectrum. Developing $F(S)$ we get the last formulas.

Corollary 3.1. *Denote $\varepsilon = 1-\delta$. The function w verifies the following Hessian equation:*

$$det(D^2w) = e_5(\Delta(w))^5 + e_3(\Delta(w))^3 S_2(w) + e_1\Delta(w)S_4(w)$$

where

$$e_5 = \frac{\varepsilon^2(168-5\varepsilon^4-24\varepsilon^3-56\varepsilon)}{(\varepsilon^2+3)(\varepsilon+7)^5(\varepsilon-2)^3}$$

$$e_3 = \frac{\varepsilon^2(2\varepsilon^2+\varepsilon+8)}{(\varepsilon-2)^2(\varepsilon+7)^3(\varepsilon^2+3)},\ e_1 = \frac{\varepsilon}{(2-\varepsilon)(\varepsilon+7)}$$

$\Delta(w) = trace(D^2w)$ being the Laplacian, $S_2(w)$ and $S_4(w)$ being respectively the second and the forth symmetric functions of the eigenvalues of D^2w.

Proof. This follows immediately from Lemma 3.1 and a simple calculation since

$$\Delta(w) = -a_{1,\delta},\ S_2(w) = a_{2,\delta},\ S_4(w) = a_{4,\delta},\ \det(D^2w) = -a_{5,\delta}.$$

Let then determine the ordered spectrum $\{\lambda_1, \lambda_2, \dots, \lambda_5\}$, $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_5$ of D^2w.

Lemma 3.2.
Let $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_5$ be the eigenvalues of $D^2w_{5,\delta}(x)$. Then

$$\lambda_1 = \mu_{2,\delta}, \quad \lambda_5 = \mu_{3,\delta},$$

$$\lambda_2 = \begin{cases} \mu_{4,\delta} & for\ p \in [-1, p_0(\delta)], \\ \mu_{1,\delta} & for\ p \in [p_0(\delta), 1], \end{cases}$$

$$\lambda_3 = \begin{cases} \mu_{5,\delta} & for\ p \in [-1, -p_0(\delta)], \\ \mu_{1,\delta} & for\ p \in [-p_0(\delta), p_0(\delta)], \\ \mu_{4,\delta} & for\ p \in [p_0(\delta), 1], \end{cases}$$

$$\lambda_4 = \begin{cases} \mu_{1,\delta} & for\ p \in [-1, -p_0(\delta)], \\ \mu_{5,\delta} & for\ p \in [-p_0(\delta), 1], \end{cases}$$

where

$$p_0(\delta) := \frac{3^{1/4}\sqrt{1-\delta}}{(3+2\delta-\delta^2)^{1/4}} = \frac{3^{1/4}\sqrt{\varepsilon}}{(4-\varepsilon^2)^{1/4}} \in]0,1].$$

Proof. The inequalities $\mu_{2,\delta}(p) \geq \mu_{1,\delta}(p) \geq \mu_{3,\delta}(p)$ are obvious since $\mu_{2,\delta}(p)$ and $\mu_{3,\delta}(p)$ are decreasing in p, $\mu_{1,\delta}(p)$ is increasing in p, $\mu_{3,\delta}(-1) = \mu_{1,\delta}(-1)$, $\mu_{2,\delta}(1) = \mu_{1,\delta}(1)$.

The resultant

$$R(\delta, p) = Res(F_2, F_3) = 144(p-1)^2(p+1)^2 \left(r_8 p^8 - r_6 p^6 + r_4 p^4 - r_2 p^2 + r_0\right)$$

where

$$r_8 = (\varepsilon^2 - 4)^2, r_6 = 12(\varepsilon^2 - 4)^2, r_4 = 3(4 - \varepsilon^2)(72 - 17\varepsilon^2),$$

$$r_2 = 108(\varepsilon^2 - 4)^2, r_0 = 144(3 - \varepsilon^2)^2$$

is strictly positive for $(\varepsilon, p) \in]0, 1[\times] - 1, 1[$. Indeed, let

$$r := \frac{R}{144(p-1)^2(p+1)^2} = r_8 p^8 - r_6 p^6 + r_4 p^4 - r_2 p^2 + r_0$$

then

$$d := \frac{\partial r}{4\varepsilon \partial \varepsilon} = (\varepsilon^2 - 4)p^8 + 12p^6(4-\varepsilon^2) + 3(17\varepsilon^2 - 70)p^4 + 108(4-\varepsilon^2)p^2 + 144(\varepsilon^2 - 3) < 0$$

for $(\varepsilon, p) \in]0, 1[\times[0, 1[$ since

$$\frac{\partial d}{4p\partial p} = (4 - \varepsilon^2)(-2p^6 + 18p^4 - 51p^2 + 54) - 6p^2 \geq (4 - \varepsilon^2) \cdot 19 - 6 \geq 51,$$

and for $p = 1$ one has $d = -166 + 76\varepsilon^2 \leq -90$. For $\delta = 0, \varepsilon = 1$ we get

$$R(\delta, p) \geq R(1, p) = 9(1 - p^2)(4 - p^2))(p^4 - 7p^2 + 16)$$

which proves the positivity. Using then the inequalities

$$\mu_{2,\delta}(-1) = \mu_{4,\delta}(-1) > \mu_{5,\delta}(-1) > \mu_{3,\delta}(-1),$$

$$\mu_{2,\delta}(1) > \mu_{4,\delta}(1) > \mu_{5,\delta}(-1) = \mu_{3,\delta}(-1)$$

and the postivity of the resultant we get

$$\mu_{2,\delta}(p) \geq \mu_{4,\delta}(p) \geq \mu_{5,\delta}(p) \geq \mu_{3,\delta}(p)$$

for any $p \in [-1, 1]$.

Calculatig then

$$R_1(\delta, p) = Res(F_1, F_3) = 12(p^2 - 3)\left(p^4(\varepsilon^2 - 4) + 3\varepsilon^2\right)$$

and taking into account the equalities

$$\mu_{4,\delta}(p_0(\delta)) = \mu_{1,\delta}(p_0(\delta)),\ \mu_{5,\delta}(-p_0(\delta)) = \mu_{1,\delta}(-p_0(\delta))$$

we get the result.

Note the oddness property of the spectrum:

$$\lambda_{1,\delta}(-p) = -\lambda_{5,\delta}(p),\ \lambda_{2,\delta}(-p) = -\lambda_{4,\delta}(p),\ \lambda_{3,\delta}(-p) = -\lambda_{3,\delta}(p).$$

Let us now verify the second condition (2.2) of Lemma 2.1, namely the uniform hyperbolicity of $M_\delta(a,b,O)$.

Proposition 3.1. *Let $M_\delta(x) = D^2 w_\delta(x),\ 0 \le \delta < 1$. Suppose that $a \neq b \in B_1 \setminus \{0\}$ and let $O \in O(5)$ be an orthogonal matrix s.t.*

$$M_\delta(a,b,O) := M_\delta(a) - {}^tO \cdot M_\delta(b) \cdot O \neq 0.$$

Denote $\Lambda_1 \ge \Lambda_2 \ge \ldots \ge \Lambda_5$ the eigenvalues of the matrix $M_\delta(a,b,O)$. Then

$$\frac{1}{C} \le -\frac{\Lambda_1}{\Lambda_5} \le C$$

for $C := \frac{1000(\delta+1)(3-\delta)}{3(1-\delta)^2}$.

Proof. The proof depends on the value of

$$k := p_0(\delta) := \frac{3^{1/4}\sqrt{1-\delta}}{(3+2\delta-\delta^2)^{1/4}} = \frac{3^{1/4}\sqrt{\varepsilon}}{(4-\varepsilon^2)^{1/4}}.$$

Note that $C = \frac{1000(\delta+1)(3-\delta)}{3(1-\delta)^2} = \frac{1000}{k^4}$. We shall give the proof for $k \in]0, \frac{1}{2}]$, the proof for $k \in [\frac{1}{2}, 1]$ is similar, simpler and uses $C = 10^4$.

Suppose that the conclusion does not hold, that is for some $a \neq b$ and some $O \in \mathrm{O}(5)$ one has

$$M_\delta(a,b,O) := M_\delta(a) - {}^tO \cdot M_\delta(b) \cdot O \neq 0,$$

but

$$\frac{1}{C} > -\frac{\Lambda_1}{\Lambda_5} \text{ or } -\frac{\Lambda_1}{\Lambda_5} > C.$$

We can suppose without loss that $|b| \le 1 = |a| \in \mathbb{S}^4_1$. Let $\overline{b} := b/|b| \in \mathbb{S}^4_1$, $W := W(a), \overline{W} := W(\overline{b}), K := |b|^{-1-\delta}$. Note that since for any harmonic cubic polynomial $Q(x)$ on $\mathbb{R}^n$ and any $a \in \mathbb{S}^{n-1}_1 \subset \mathbb{R}^n$ one has

$$Tr\left(D^2\left(\frac{Q(x)}{|x|^{1+\delta}}\right)(a)\right) = (\delta^2 - 2\delta - 3 - n)Q(a),$$

we get $Tr\left(M_\delta(a,b,O)\right) = (2\delta + 8 - \delta^2)(K\overline{W} - W),\ P_5$ being harmonic. Let us prove the claim for $(K\overline{W} - W) \ge 0$, the proof for $(K\overline{W} - W) \le 0$ being the same while permuting a with b and Λ_1 with Λ_5. Since

$$Tr\left(M_\delta(a,b,O)\right) = (2\delta + 8 - \delta^2)(K\overline{W} - W) \ge 0,$$

we get $4\Lambda_1 + \Lambda_5 \geq 0$ and $-\Lambda_5/\Lambda_1 \leq 4$. Therefore, we have only to rule out the inequality $\frac{1}{C} > -\frac{\Lambda_1}{\Lambda_5}$ i.e. $-\Lambda_5 > C\Lambda_1$. Recall that

$$W = \frac{3p - p^3}{2}, \overline{W} = \frac{3\overline{p} - \overline{p}^3}{2}$$

for some $p, \overline{p} \in [-1, 1]$.

We have then 3 possibilities:

1). $p, \overline{p} \in [-k, k]$;

2). $p \in [-k, k], \overline{p} \notin [-k, k]$;

3). $p, \overline{p} \notin [-k, k]$.

In the cases 1) and 3) applying Lemma 2.4 we get $\Lambda_1 \geq \mu_+(K)$, $\Lambda_5 \leq \mu_-(K)$ in the notation of Lemma 2.5 which permits to finish the proof as in Proposition 4.1 of [NV3]. We thus have to treat the (more difficult) case 2). Lemma 2.4 together with the inequality $-\Lambda_5 > C\Lambda_1$ gives

$$-\min_{i=1,\cdots,5}\{K\lambda_{i,\delta}(\overline{p}) - \lambda_{i,\delta}(p)\} > C \max_{i=1,\cdots,5}\{K\lambda_{i,\delta}(\overline{p}) - \lambda_{i,\delta}(p)\}.$$

Thank to the oddness of the spectrum we suppose without loss that $\overline{p} > k$. Recall that then by Lemma 3.2 one has

$$\lambda_{1,\delta}(\overline{p}) = \mu_{2,\delta}(\overline{p}), \lambda_{1,\delta}(p) = \mu_{2,\delta}(p),\ \lambda_{2,\delta}(\overline{p}) = \mu_{1,\delta}(\overline{p}), \lambda_{1,\delta}(p) = \mu_{4,\delta}(p),$$

$$\lambda_{3,\delta}(\overline{p}) = \mu_{4,\delta}(\overline{p}), \lambda_{3,\delta}(p) = \mu_{1,\delta}(p),\ \lambda_{4,\delta}(\overline{p}) = \mu_{5,\delta}(\overline{p}),\ \lambda_{4,\delta}(p) = \mu_{5,\delta}(p),$$
$$\lambda_{5,\delta}(\overline{p}) = \mu_{3,\delta}(\overline{p}),\ \lambda_{5,\delta}(p) = \mu_{3,\delta}(p).$$

We have then 2 possibilities for p :

2a). $p \in [-k, 0]$;

2b). $p \in]0, k]$.

Let $p \in [-k, 0]$, then $\mu_{1,\delta}(p) \leq 0$ and thus

$$C \max_{i=1,\cdots,5}\{K\lambda_{i,\delta}(\overline{p}) - \lambda_{i,\delta}(p)\} \geq CK\lambda_{3,\delta}(\overline{p}) = CK\mu_{4,\delta}(\overline{p}) \geq CK\mu_{4,\delta}(p_0(\delta)) =$$

$$= CK\mu_{1,\delta}(k) = CK(k^3(\sqrt{k^4+3} - k^2 + 3)/\sqrt{k^4+3}) \geq 2CKk^3$$

since one verifies that the function $\mu_{4,\delta}(p)$ is increasing on $[k, 1]$.

On the other hand,

$$|\min_{i=1,\cdots,5}\{K\lambda_{i,\delta}(\overline{p}) - \lambda_{i,\delta}(p)\}| \leq K \max_{i=1,\cdots,5,p} |\{\lambda_{i,\delta}(p)\}| + \max_{i=1,\cdots,5,p} |\{\lambda_{i,\delta}(p)\}| \leq 8(K+1).$$

Therefore one gets $8(K+1) \geq 2CKk^3$ which clearly is a contradiction for, say, $K \geq 1/4$. For $0 < K \leq 1/4$ we get

$$C \max_{i=1,\cdots,5}\{K\lambda_{i,\delta}(\overline{p}) - \lambda_{i,\delta}(p)\} \geq C(K\lambda_{5,\delta}(\overline{p}) - \lambda_{5,\delta}(p)) \geq C(K(-8) - (-5)) \geq 3C$$

which can not be less than $8(K+1) \leq 10$.

Let finally $p \in]0, k]$. We consider then 2 possibilities for K:

(i) $K \leq 20/31 = (1.55)^{-1}$,

(ii) $K > 20/31 = (1.55)^{-1}$.

In the case (i) one has

$$C \max_{i=1,\cdots,5} \{K\lambda_{i,\delta}(\overline{p}) - \lambda_{i,\delta}(p)\} \geq C(K\lambda_{5,\delta}(\overline{p}) - \lambda_{5,\delta}(p)) \geq$$

$$\geq C(K\mu_{3,\delta}(1) - \mu_{3,\delta}(0)) \geq C(3\sqrt{3} + 20(\varepsilon - 8)/31) > C/30 > 8(K+1)$$

since $\lambda_{5,\delta}(p) = \mu_{3,\delta}(p)$ is decreasing, $\mu_{3,\delta}(0) = -3\sqrt{3}, \mu_{3,\delta}(1) = \varepsilon - 8$.

We suppose then $K > 20/31 = (1.55)^{-1}$. Then if $p \leq 3k/4$ one has

$$C \max_{i=1,\cdots,5} \{K\lambda_{i,\delta}(\overline{p}) - \lambda_{i,\delta}(p)\} \geq CK(\lambda_{3,\delta}(\overline{p}) - \lambda_{3,\delta}(p)/K) \geq CK(\mu_{4,\delta}(\overline{p}) - \mu_{1,\delta}(p)/K) \geq$$

$$\geq CK(\mu_{4,\delta}(k) - \mu_{1,\delta}(3k/4)/K) = CK(\mu_{1,\delta}(k) - \mu_{1,\delta}(3k/4)/K) >$$

$$> CK\mu_{1,\delta}\left(\frac{3k}{4}\right)\left(\frac{\mu_{4,\delta}(k))}{\mu_{1,\delta}\left(\frac{3k}{4}\right)} - \frac{31}{20}\right) \geq$$

$$\geq CK\frac{\mu_{1,\delta}\left(\frac{3k}{4}\right)}{100} \geq CK\frac{2k^3}{100} > 20K > 8(K+1),$$

contradiction for $k \in [0, 1/2]$ since $\frac{\mu_{4,\delta}(k))}{\mu_{1,\delta}\left(\frac{3k}{4}\right)} > 1.56, \mu_{1,\delta}\left(\frac{3k}{4}\right) > 2k^3$ there. Thus we can suppose that $p \in]\frac{3k}{4}, k]$. One notes then that $\mu_{4,\delta}(p) \leq \mu_{4,\delta}(k)$ for $p \in [\frac{k}{4}, 1]$. This permits to rule out the case $\overline{p} \geq \frac{3k}{2}$. Indeed, one has in this case

$$C \max_{i=1,\cdots,5} \{K\lambda_{i,\delta}(\overline{p}) - \lambda_{i,\delta}(p)\} \geq CK(\lambda_{2,\delta}(\overline{p}) - \lambda_{2,\delta}(p)/K) \geq CK(\mu_{1,\delta}(\overline{p}) - \mu_{4,\delta}(p)/K) \geq$$

$$\geq CK(\mu_{1,\delta}(3k/2) - \mu_{1,\delta}(k)/K) = CK(\mu_{1,\delta}(3k/2) - \mu_{1,\delta}(k)/K),$$

and one gets a contradiction as above since

$$\frac{\mu_{1,\delta}\left(\frac{3k}{2}\right)}{\mu_{1,\delta}(k)} > 2$$

for $k \in [0, 1/2]$.

The last case to rule out is thus $K \geq 20/31, p \in [\frac{3k}{4}, k], \overline{p} \in [k, \frac{3k}{2}]$. Let then

$$\alpha := k - p \in [0, \frac{k}{4}] \subset [0, \frac{1}{8}],\ \beta := \overline{p} - k \in [0, \frac{k}{2}] \subset [0, \frac{1}{4}],\ a := \max\{\alpha, \beta\}.$$

It is easy to verify that on $\left[\frac{3k}{4}, \frac{3k}{2}\right]$ one has the following inequalities:

$$\frac{\partial \mu_{1,\delta}(p)}{\partial p} \geq 3k^2,\ \frac{11k^3}{4} \geq \mu_{1,\delta}(k) \geq \frac{5k^3}{2}\ ;$$

$$\frac{\partial \mu_{2,\delta}(p)}{\partial p} \geq -5,\ 4k-5 \geq \mu_{2,\delta}(k) \geq 4k - \frac{11}{2}\ ;$$
$$\frac{\partial \mu_{3,\delta}(p)}{\partial p} \geq -\frac{9}{2},\ -5-3k \geq \mu_{3,\delta}(k) \geq -\frac{11+7k}{2}\ ;$$
$$\frac{\partial \mu_{4,\delta}(p)}{\partial p} \geq -\frac{k}{29},\ \frac{11k^3}{4} \geq \mu_{4,\delta}(k) \geq \frac{5k^3}{2}\ ;$$
$$\frac{\partial \mu_{5,\delta}(p)}{\partial p} \geq 10k^2-12,\ -10k \geq \mu_{5,\delta}(k) \geq -12k\ .$$

Let then $K \in \left[\frac{20}{31}, 1\right]$. Therefore,

$$C \max_{i=1,\cdots,5}\{K\lambda_{i,\delta}(\overline{p}) - \lambda_{i,\delta}(p)\} \geq$$
$$\geq C\max\{K\mu_{1,\delta}(k+\alpha)-\mu_{1,\delta}(k-\beta), K\mu_{3,\delta}(k+\alpha)-\mu_{3,\delta}(k-\beta)\} \geq \max\{M_1, M_2\} =$$
$$= C\max\left\{3(K-1)k^3a + 3(K+1)k^2, (1-K)(5+3k)a - \frac{(11+7k)(K+1)}{10}\right\}$$

for linear forms M_1, M_2 in K. Note that the minimal value of $\max\{M_1, M_2\}$ as a function of K equals (recall that our $C = 1000/k^4$):

$$\frac{1500a(40-9k)}{k^2(12k^2a+18a+11k^3+20+12k)} > \frac{1250a}{k^2} > 0$$

attained for $K = K_0 := (11k^3+20+12k)/(12k^2a+18a+11k^3+20+12k) < 1$.

On the other hand,

$$-\Lambda_5 \leq -\min_{i=1,\cdots,5}\{K\lambda_{i,\delta}(\overline{p}) - \lambda_{i,\delta}(p)\} \leq \max\{l_1, l_2, l_3, l_4, l_5\}$$

for the following linear forms (in K)

$$l_1 := -k^2\left(\frac{11a}{4} + 3k\right)K - \frac{11a}{4}k^2 + 3k^3,$$
$$l_2 := \left(5a + 4k - \frac{11}{2}\right)K + 5a + \frac{11}{2} - 4k,$$
$$l_3 := (5a+5+3k)K + 5a - 5 - 3k,$$
$$l_4 := \left(\frac{ak}{29} - \frac{11k^3}{2}\right)K + \frac{ak}{29} + \frac{11k^3}{2},$$
$$l_5 := ((12-10k^2)a + 12k)K + (12-10k^2)a - 12k.$$

To refute our inequality it is sufficient to prove that $M_i(K_{j,k}) > 0$ for any triple (i,j,k) with $i,j \in \{1,2\},\ i \neq j,\ k \in \{1,2,3,4,5\}$ where $l_k(K_{j,k}) = M_j(K_{j,k})$.

Explicit calculations give (for the values $m_{ijk} := \frac{M_i(K_{j,k})}{500ak^2}$)

$$\frac{m_{121}}{3} = \frac{(9k^4+6k^6)a + 10000 + 5k^7 + 20k^4 + 3k^5 - 2250k}{k^2((3k^4+3000)a + 3k^5 + 2750k)} > 0,$$

$$\frac{m_{211}}{3} = \frac{(9k^4+6k^6)a+10000+5k^7+20k^4+3k^5-2250k}{(4500-3k^6)a+5000+3000k-3k^7} > 0,$$

$$m_{122} = \frac{60000-(60k^4+90k^2)a-192k^3+66k^4-13500k-101k^2-158k^5}{k^2((6000-10k^2)a-11k^2+8k^3+5500k)} > 0,$$

$$m_{212} = \frac{60000-(60k^4+90k^2)a-192k^3+66k^4-13500k-101k^2-158k^5}{(10k^4+9000)a+11k^4-8k^5+10000+6000k} > 0,$$

$$m_{123} = \frac{30000-(45k^2+30k^4)a-55k^2-6750k-33k^3+30k^4-37k^5}{k^2((3000-5k^2)a+2750k-5k^2-3k^3)} > 0,$$

$$m_{213} = \frac{30000-(45k^2+30k^4)a-55k^2-6750k-33k^3+30k^4-37k^5}{(5k^4+4500)a+5k^4+3k^5+5000+3000k} > 0,$$

$$m_{124} = \frac{3480000-(36k^3+24k^5)a-t(k)}{k^2((348000-4k^3)a+319000k+319k^5)} > 0,$$

$$m_{214} = \frac{3480000-(36k^3+24k^5)a-t(k)}{(522000+4k^5)a+580000+348000k-319k^7} > 0,$$

$$m_{215} = \frac{(9k^4+30k^6-54k^2)a+15000-u(k)}{5k^4a+1500a-6k^3+1375k-6k^2a} > 0,$$

$$m_{125} = \frac{(9k^4+30k^6-54k^2)a+15000-u(k)}{(2250+6k^4-5k^6)a+2500+1500k+6k^5} > 0,$$

where $t(k) := 783000k+80k^3+48k^4+44k^6+2871k^5+1914k^7 < 4\cdot 10^5$, $u(k) := 120k^2-30k^5+3375k+18k^3-100k^4-55k^7 < 2000$.

Let, finally $K \geq 1$, then

$$C\Lambda_1 \geq C(K\mu_{1,\delta}(k+\alpha)-\mu_{1,\delta}(k-\beta)) \geq \frac{5C}{2}(K-1)k^3a+3(K+1)Ck^2 =$$

$$= L_1 := \frac{500(5kK+6a-5k)}{k^2} = \frac{2500(K-1)}{k}+\frac{3000a}{k^2},$$

and

$$-\Lambda_5 \leq -\min_{i=2,\cdots,5}\{K\lambda_{i,\delta}(\overline{p})-\lambda_{i,\delta}(p)\} \leq \max\{L_2,L_3,L_4,L_5\}$$

for the following linears forms in K:

$$L_2 := 5(K+1)a+(1-K)(5-4k) = (5a-5+4k)(K-1)+10a,$$

$$L_3 := \frac{9}{2}(K+1)a+\frac{11+7k}{2}(K-1) = \frac{9a+11+7k}{2}(K-1)+9a,$$

$$L_4 := (K+1)a\frac{k}{29}-\frac{5k^3}{2}(K-1) = \left(\frac{ak}{29}-\frac{5k^3}{2}\right)(K-1)+\frac{2ak}{29},$$

$$L_5 := 12k(K+1)a+\frac{9}{2}(K-1) = 3\left(4ak+\frac{3}{2}\right)(K-1)+24ak.$$

One immediately sees that both the slope and the value at $K = 1$ of L_1 are (much) bigger than those of $L_i, i = 2, 3, 4, 5$ which finishes the proof.

Proof of Theorem 1.1. To prove the result it is sufficient to verify the condition (2.1) in Lemma 2.1, namely, that the five partial derivatives $\frac{\partial g}{\partial \lambda_i}, i = 1, \ldots, 5$ are strictly positive (and bounded which is automatic thank to compacity) on the symmetrized image

$$M := \bigcup_{\sigma \in \Sigma_n} T_\sigma \Lambda(B) \subset \mathbb{R}^n$$

of the unit ball under the map Λ,

$$g(\lambda_1, \ldots, \lambda_5) = det(D^2 w) - e_5(\Delta(w))^5 - e_3(\Delta(w))^3 S_2(w) - e_1 \Delta(w) S_4(w)$$

being our equation. By homogeneity it is sufficient to show this on $M' := \Lambda(\mathbb{S}^4_1)$ which is an algebraic curve, the union of 120 curves $T_\sigma \Lambda(\mathbb{S}^4_1)$ and that it is sufficient, by symmetry, to verify the condition on the curve $\Lambda(\mathbb{S}^4_1)$ only. A brute force calculation shows then that

$$g_1(p, \varepsilon) := \frac{\partial g}{\partial \lambda_1} = \sum_{i=0}^{12} m_i p^i =$$

$$= m_{12} p^8 (p^4 - 12p^2 + 54) + m_9 b^3 + m_6 p^4 (p^2 - \frac{3}{4}) + m_2 + m_0,$$

with

$$m_{12} = 3(\varepsilon^4 + 3\varepsilon^3 - 20\varepsilon^2 + 12\varepsilon - 56)(\varepsilon - 2)^2 (\varepsilon + 2)^2, m_{10} = -12 m_{12}, m_8 = 54 m_{12},$$

$$m_{11} = m_1 = 0, m_9 = 3D(p, \varepsilon)(\varepsilon + 7)(\varepsilon + 2)(\varepsilon^2 + 2)(\varepsilon - 2)^2, m_7 = -9 m_9, m_5 = 27 m_9,$$

$$m_6 = 108(2 - \varepsilon)(3\varepsilon^7 + 17\varepsilon^6 - 54\varepsilon^5 - 152\varepsilon^4 + 72\varepsilon^3 - 42\varepsilon^2 + 384\varepsilon + 1344), m_3 = 27 m_9,$$

$$m_4 = -\frac{3 m_6}{4}, \;\; m_2 = 1944\varepsilon^2 (2 - \varepsilon)(\varepsilon^2 - 7)(\varepsilon^2 + 3), \;\; m_0 = -7776\varepsilon^2 (\varepsilon^2 + 3)$$

for $D(p, \varepsilon) := \sqrt{(16 - \varepsilon^2)(4 - \varepsilon^2) b^2 + 144\varepsilon^2}$, $b = (p^2 - 3)p$;

$$g_2(p, \varepsilon) := \frac{\partial g}{\partial \lambda_2} = \sum_{i=0}^{12} n_i p^i$$

with

$$n_{12} = (\varepsilon + 4)(\varepsilon + 1)(4 - \varepsilon^2)^2, n_{10} = -(\varepsilon + 2)(\varepsilon^4 + 19\varepsilon^3 + 86\varepsilon^2 + 182\varepsilon + 96)(2 - \varepsilon)^2,$$

$$n_{11} = n_1 = 0, \;\; n_8 = 9(\varepsilon + 2)(\varepsilon^2 + 10\varepsilon + 6)(\varepsilon^2 + 3\varepsilon + 8)(2 - \varepsilon)^2,$$

$$n_9 = \varepsilon(\varepsilon + 7)(\varepsilon + 2)(\varepsilon^2 + 2)(2 - \varepsilon)^2 \sqrt{3(4 - p^2)}, \; n_7 = -9 n_9, \; n_5 = 27 n_9,$$

$$n_6 = 3(2 - \varepsilon)(13\varepsilon^6 + 115\varepsilon^5 + 218\varepsilon^4 + 170\varepsilon^3 - 876\varepsilon^2 - 2856\varepsilon - 1152),$$

$$n_4 = 9(\varepsilon - 2)(11\varepsilon^6 + 62\varepsilon^5 + 33\varepsilon^4 - 24\varepsilon^3 - 348\varepsilon^2 - 1176\varepsilon - 288),$$
$$n_3 = \varepsilon^3(2-\varepsilon)(\varepsilon+7)(3\varepsilon^2+2)\sqrt{3(4-p^2)},$$
$$n_2 = 108\varepsilon^2(2-\varepsilon)(\varepsilon^2+3)(\varepsilon^2+4\varepsilon-3), n_0 = 1296\varepsilon^2(\varepsilon^2+3);$$

$$g_3(p,\varepsilon) := \frac{\partial g}{\partial \lambda_3} = \sum_{i=0}^{6} h_{2i}p^{2i}$$

with

$$h_{12} = (\varepsilon+4)(\varepsilon+1)(4-\varepsilon^2)^2, h_{10} = 2(\varepsilon+2)(\varepsilon^4+\varepsilon^3-40\varepsilon^2-70\varepsilon-48)(2-\varepsilon)^2$$
$$h_8 = -18(\varepsilon+2)(\varepsilon^4+4\varepsilon^3-19\varepsilon^2-28\varepsilon-24)(2-\varepsilon)^2,$$
$$h_6 = 6(\varepsilon-2)(7\varepsilon^6+37\varepsilon^5-136\varepsilon^4-274\varepsilon^3+330\varepsilon^2+672\varepsilon+576),$$
$$h_4 = 9(\varepsilon-2)(2\varepsilon^6-\varepsilon^5+27\varepsilon^4-66\varepsilon^3-348\varepsilon^2-1176\varepsilon-288),$$
$$h_2 = 108\varepsilon(2-\varepsilon)(\varepsilon^3+4\varepsilon^2-15\varepsilon-84)(\varepsilon^2+3),$$
$$h_0 = -1296\varepsilon(\varepsilon^2+3)(\varepsilon^2+4\varepsilon-14);$$

$$g_4(p,\varepsilon) := \frac{\partial g}{\partial \lambda_4} = g_1(-p,\varepsilon),$$
$$g_5(p),\varepsilon := \frac{\partial g}{\partial \lambda_5} = g_2(-p,\varepsilon),$$

and thus we need to consider only the functions $g_1(p,\varepsilon), g_2(p,\varepsilon), g_3(p,\varepsilon)$ on the set $[-1,1]\times(0,1]$. We have to prove that for any fixed $\varepsilon \in (0,1]$ they are strictly positive.

The technique of the proof is identical for all three derivatives, and we begin with g_3 wich is slightly simpler since it is a polynomial in two variables. One can rearrange it in the form

$$g_3(p,\varepsilon) = g_{37}\varepsilon^7 + g_{36}\varepsilon^6 + g_{35}\varepsilon^5 + g_{34}\varepsilon^4 + g_{33}\varepsilon^3 + g_{32}\varepsilon^2 + g_{31}\varepsilon + g_{30}$$

with

$$g_{37} = 2q^5+42q^3-18q^4-108q+18q^2,\ g_{36} = 138q^3-2q^5-216q+q^6-36q^4-45q^2,$$
$$g_{35} = -92q^5+558q^4+2160q+5q^6-1296-1260q^3+261q^2,$$
$$g_{34} = -4q^6+5184q-12q^3-5184-1080q^2+28q^5-36q^4,$$
$$g_{33} = -1944q^2-2520q^4-40q^6+14256-10692q+5268q^3+520q^5,$$
$$g_{32} = -144q^4+72q^3+112q^5+17496q-4320q^2-16q^6-15552,$$
$$g_{31} = -736q^5+80q^6+2304q^4-54432q+54432-4608q^3+18576q^2, g_{30} = 64q^2(q-3)^4 \geq 0$$

for $q = p^2 \in [0,1]$. Therefore,

$$g_3(p,\varepsilon) \geq \varepsilon(\bar{g}_{37}\varepsilon^6 + \bar{g}_{36}\varepsilon^5 + \bar{g}_{35}\varepsilon^4 + \bar{g}_{34}\varepsilon^3 + \bar{g}_{33}\varepsilon^2 + \bar{g}_{32}\varepsilon + \bar{g}_{31})$$

where $\bar{g}_{3i} := \min_{q\in[0,1]} g_{3i}(q)$, and an elementary calculation gives

$$\frac{g_3(p,\varepsilon)}{\varepsilon} \geq -64\varepsilon^5 - 160\varepsilon^4 - 5184\varepsilon^3 + 4848\varepsilon^2 - 15552\varepsilon + 15616 > 1620$$

for $\varepsilon \in (0, \frac{9}{10}]$. For $\varepsilon \in [\frac{9}{10}, 1]$ we have $g_3(p,\varepsilon) \geq \sum_{i=0}^{6} \bar{h}_{2i} q^i$ where $\bar{h}_{2i} := \min_{\varepsilon\in[\frac{9}{10},1]} h_{2i}$ and thus

$$g_3(p,\varepsilon) \geq -736q^5 + 80q^6 + 2304q^4 - 54432q + 54432 - 4608q^3 + 18576q^2 > 4840.$$

Thus, finally $g_3(p,\varepsilon) \geq \min\{1620\varepsilon, 4840\}$.

The function $g_1(p,\varepsilon) = s_1 - t_1$ with

$$\begin{gathered}
s_1 := (q^3 - 6q^2 + 9q)\varepsilon^7 + (5q^3 - 30q^2 + 45q - 72)\varepsilon^6 + (108q^2 - 18q^3 - 162q)\varepsilon^5 + \\
(-432q + 288 - 48q^3 + 288q^2)\varepsilon^4 + (24q^3 - 144q^2 + 216q)\varepsilon^3 + \\
1512\varepsilon^2 + (1152q - 768q^2 + 128q^3)\varepsilon + 448q^2(q-3)^4 \geq \\
-72\varepsilon^6 - 72\varepsilon^5 + 96\varepsilon^4 + 1512\varepsilon^2 \geq 1440\varepsilon^2 > 0,
\end{gathered}$$

and

$$t_1 := (\varepsilon^2 - 4)(\varepsilon + 7)(\varepsilon^2 + 2)bD(p,\varepsilon);$$

simplifying $s_1^2 - t_1^2 = (s_1 - t_1)(s_1 + t_1) = g_1(p,\varepsilon)(s_1 + t_1)$ one finds

$$\begin{gathered}
(540q^2 - 1296q - 216q^4 - 4q^6 + 288q^3 + 48q^5)\varepsilon^{13} + \\
(-1944q^4 + 3024q^3 + 432q^5 - 7776q + 5184 + 2268q^2 - 36q^6)\varepsilon^{12} + \\
(-2052q^2 + 3240q^4 - 5328q^3 - 720q^5 + 60q^6 + 10368q)\varepsilon^{11} + \\
(936q^6 + 50544q^4 + 55944q^2 - 11232q^5 - 41472 - 97776q^3 + 29808q)\varepsilon^{10} \\
+(26136q^2 + 120q^6 - 15696q^3 + 6480q^4 - 24624q - 1440q^5)\varepsilon^9 + \\
(57456q^5 + 156816q - 492372q^2 - 4788q^6 - 134784 + 534528q^3 - 258552q^4)\varepsilon^8 + \\
(180576q^2 - 352q^6 - 313632q + 3168q^3 + 4224q^5 - 19008q^4)\varepsilon^7 + \\
(54432q^4 - 238464q^3 + 859248q^2 - 1166400q + 1008q^6 + 870912 - 12096q^5)\varepsilon^6 + \\
(1118592q^3 - 518400q^4 - 9600q^6 - 1268352q^2 + 736128q + 115200q^5)\varepsilon^5 + \\
(1524096q^4 + 207360q + 2286144 + 28224q^6 - 338688q^5 + 2147904q^2 - 3025152q^3)\varepsilon^4 + \\
(10752q^6 + 580608q^4 - 903168q^3 - 677376q^2 + 2322432q - 129024q^5)\varepsilon^3 + \\
(3640320q^3 - 25344q^6 + 304128q^5 - 1368576q^4 + 8128512q - 7471872q^2)\varepsilon^2 + \\
(3096576q^4 + 4644864q^2 + 57344q^6 - 688128q^5 - 6193152q^3)\varepsilon \geq
\end{gathered}$$

$$64\varepsilon^4(-10\varepsilon^9+18\varepsilon^8-648\varepsilon^6-141\varepsilon^5-2214\varepsilon^4-2266\varepsilon^3+5760\varepsilon^2+35721) \geq 2.2\cdot 10^6\varepsilon^4.$$

Since $s_1 + t_1 \leq 10^6$ one gets $g_1(p, \varepsilon) > 2\varepsilon^4$.

Similarly, $g_2(p, \varepsilon) = s_2 - t_2$ with a polynomial $s_2 \geq 3000\varepsilon^2$ and

$$t_2 = \varepsilon(\varepsilon + 7)(\varepsilon - 2)t(\varepsilon, q)p^3\sqrt{3(4 - q)}$$

where

$$t(\varepsilon, q) := (\varepsilon^4 - 2\varepsilon^2 - 8)q^3 + 9(-\varepsilon^4 + 2\varepsilon^2 + 8)q^2 + 27(\varepsilon^4 - 2\varepsilon^2 - 8)p^2 - 9(3\varepsilon^2 + 2)\varepsilon^2.$$

Simplifying $s_2^2 - t_2^2 = g_2(p, \varepsilon)(s_2 + t_2)$ one gets a polynomial $\geq$

$$(-2560\varepsilon^9 - 18176\varepsilon^8 - 325632\varepsilon^6 - 1254656\varepsilon^4 + 2202112\varepsilon^2 + 15116544)\varepsilon^4 \geq 1.5\cdot 10^7\varepsilon^4$$

and $g_2(p, \varepsilon) \geq 15\varepsilon^4$ which finishes the proof.

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